Journal of Pure and Applied Algebra 215 (2011) 2628–2638

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

journal homepage: www.elsevier.com/locate/jpaa

The Euler Class in homological algebra

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ARTICLE INFO

Article history:
Received 27 January 2011
Available online 26 March 2011
Communicated by E.M. Friedlander

MSC: 55R40; 55U10

ABSTRACT

Let G_τ be the topological group of orientation preserving homeomorphisms of the circle, and G_δ the same group with the discrete topology. Motivated by the classical problem of reducing a circle bundle with structure group G_τ to a totally disconnected subgroup $K \subset G_\delta$, and more currently, applications to mapping class groups, we analyze, in a homological algebra setting, the role played by the Topological and Discrete Euler Classes. In particular we describe the Discrete Euler Class of G, and any of its subgroups K, explicitly as a group extension. We apply our constructions to show that the values of the Discrete Euler Class are bounded on any space, and we state triviality and non-triviality conditions for its powers in the based mapping class groups.

1. Introduction

Let G_τ be the topological group of orientation preserving homeomorphisms of the circle, and G_δ the same group with the discrete topology. As is well known, the classifying space BG_τ is complex projective space and the Classical Euler Class $[E_\tau] \in H^2(BG_\tau, \mathbb{Z})$ generates its cohomology ring. Not as well known is the fact that the identity $G_\delta \to G_\tau$ induces a homology equivalence $BG_\delta \to BG_\tau$, so that Classical Euler Class pull backs to a non-vanishing class, which we refer to as the Discrete Euler Class. Consider the classical problem of reducing a circle bundle with structure group G_τ to a totally disconnected subgroup $K \subset G_\delta$. What the homology equivalence makes apparent is that from an algebraic standpoint a deep aspect of the reduction problem involves the relationship between the cohomology of G_δ and that of its subgroups.

In this work we give a self-contained treatment, in the setting of homological algebra, of the Discrete and Topological Euler Classes. We begin with a new description of the Discrete Euler Class of G as a group extension $0 \to \mathbb{Z} \to G_- \to G \to 0$. The lifting problem for this extension involves choosing coherently left and right branches of homeomorphisms of the circle when they are considered as homeomorphisms of the reals. We then give a direct proof, using simplicial techniques, of the homological equivalence $BG_\delta \to BG_\tau$, [10,14]. The goal here is twofold – to give an elementary proof in our setting, and to show explicitly how the Discrete and Topological Euler Classes match up – to each other and to that defined by the extension.

We conclude with applications. First we show that the values of the Discrete Euler Class are bounded on a topological space, generalizing a theorem of Wood, [15]. A version of this result appeared in [6], but the treatment we give here is more direct—an easy consequence of the cocycle formula for the extension.

The Based Mapping Class Groups, $\mathcal{M}_{g,*}$, can be identified with subgroups of G_δ, [2,12], and so the Discrete Euler Class is an invariant. Morita conjectured that, over the rationals, the powers vanish at dimension g and beyond. We present results, proved in [7,8], which imply that, with integer coefficients, $[E^n_{\mathcal{M}_{g,*}}]$ determines a non-zero homomorphism to $\mathbb{Z}$ for $n < g$, $[E^n_{\mathcal{M}_{g,*}}] = 0$, for $n > g$, and at the threshold dimension g, the class $[E^g_{\mathcal{M}_{g,*}}]$ has torsion $2g(2g+1)$.

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doi:10.1016/j.jpaa.2011.03.005

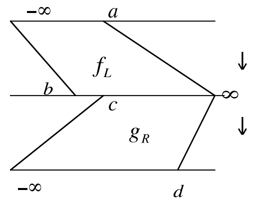


Fig. 1. Left–right composition.

2. The Euler Class as an extension: branched homeomorphisms of the circle

In this section let $G = G_\delta$ be the discrete group of orientation preserving homeomorphism of the circle $S^1 = \mathbb{R} \cup \infty$. Suppose $f \in G$ maps ∞ to b and a to ∞. If $\infty \neq a$ then f determines two homeomorphisms: a left branch, $f_L : (-\infty, a) \to (b, \infty)$, and a right branch $f_R : (a, \infty) \to (-\infty, b)$. If $\infty = a$, or $\infty = b$, then $f_L = f_R$. Suppose $f, g \in G$ determine the branches

$$f_L : (-\infty, a) \to (b, \infty), \quad \text{and} \quad f_R : (a, \infty) \to (-\infty, b)$$
$$g_L : (-\infty, c) \to (d, \infty), \quad \text{and} \quad g_R : (c, \infty) \to (-\infty, d).$$

Then the following relations are satisfied.
$g_L \circ f_L = (g \circ f)_L$ if the composite is defined.
$g_R \circ f_R = (g \circ f)_R$ if the composite is defined.
$g_R \circ f_L = (g \circ f)_R$ if $b \leq c$ and $g_R \circ f_L = (g \circ f)_L$ if $b \geq c$.
$g_L \circ f_R = (g \circ f)_R$ if $b \leq c$ and $g_L \circ f_R = (g \circ f)_L$ if $b \geq c$.
$(f_L)^{-1} = (f^{-1})_R$.
$(f_R)^{-1} = (f^{-1})_L$.

Fig. 1 illustrates the property $g_R \circ f_L = (g \circ f)_R$ if $b \leq c$. If b is strictly less than c the composite is not defined at $-\infty$, so it must be defined at $+\infty$, hence the composite is the right branch. If $b = c$ the composite is defined at $-\infty$, but maps $-\infty$ to itself. In this case the composite must also map $+\infty$ to itself so that $(g \circ f)_R = (g \circ f)_L$.

The other properties are verified in a similar manner.

The composition $g_- \circ f_-$ of two branches g_- and f_- is defined if and only if the range of f_- intersects the domain of g_-. This partially defined multiplication makes the set of all branches $G_L \cup G_R$ of G into a pregroup, [13]. The definition follows.

Definition. A *pregroup*, [13], consists of a set S containing a distinguished element 1, each element $s \in S$ has a unique inverse s^{-1} and to each pair of elements $s, t \in S$ there is defined at most one product $st \in S$ such that

(a) $1s = s1 = s$ always defined.
(b) $ss^{-1} = s^{-1}s = 1$ always defined.
(c) If st is defined then $t^{-1}s^{-1}$ is defined and equal to $(st)^{-1}$.
(d) If rs and st are defined then $r(st)$ is defined if and only if $r(st)$ is defined, in which case the two are equal.
(e) If qr, rs and st are defined then either $q(rs)$ is defined or $r(st)$ is defined.

The universal group of a pregroup is the free group on all its elements modulo the relations $x \cdot y = x \circ y$ whenever the composition on the right is defined. In the universal group of a pregroup every generator, other than the identity, represents a distinct non-trivial element. Moreover every minimal word, that is one in which any two successive generators are incomposable, represents a non-trivial element in the universal group. In [5], a more general notion of pregroup, one that replaces conditions (a)–(e) by a weak associativity condition, is defined. With either set of assumptions, it is shown in [5] that all minimal words, other than 1 are non-trivial.

We denote the universal group of $G_L \cup G_R$ by G_-, and the corresponding extension by E. The identity element of the universal group is the branch of the identity homeomorphism. The inverse of f_L is $(f^{-1})_R$.

We prove the following theorem.

Theorem 1. *The characteristic cohomology class of the extension* $0 \to A \to G_- \to G \to 1$ *is the Discrete Euler Class of* G.

The proof will only be complete after we have explicitly identified the characteristic class of the extension to the Topological Euler Class, §5. The first step in the proof is the following.

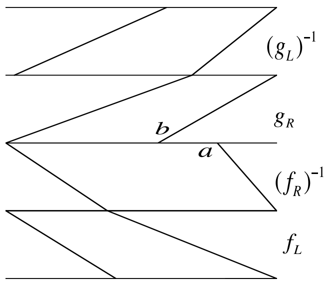


Fig. 2. Products in K_-.

Lemma A. *The kernel A of the natural map $G_- \to G$ is isomorphic to the integers $\mathbb{Z}$.*

Proof of Lemma A. Let $x = f_- \cdot g_- \cdots$ be a word in the branches of G which is minimal, that is no two successive generators are composable. A left and a right branch are always composable, so a minimal word consists entirely of left branches or entirely of right branches. Let us assume that it has left branches. If the word x begins with $f_L \cdot g_L$ then inserting the trivial word $((f \circ g)^{-1})_L \cdot (f \circ g)_R$ directly after it gives $\vec{x} = f_L \cdot g_L \cdot ((f \circ g)^{-1})_L \cdot (f \circ g)_R \cdots$. The generator $(f \circ g)_R$ can be composed with at least one branch to its right, since a right and a left are always composable. Carrying out such compositions until there are only left branches remaining, repeating the procedure as often as necessary, and arguing similarly with a minimal word made up of right branches leads to:

Any word in A can be written as a product of elements each one of which is either of the form $f_L \cdot g_L \cdot ((f \circ g)^{-1})_L$, or of the form $f_R \cdot g_R \cdot ((f \circ g)^{-1})_R$.

Note that such a word formed by right branches is the inverse of such a word formed by left branches.

So, consider $f_L \cdot g_L \cdot ((f \circ g)^{-1})_L$ and assume that $f_L \cdot g_L$ is minimal, which means that $g(-\infty) \geq f^{-1}(\infty)$. Since $(f^{-1})_L \cdot f_R$ is the identity the word $f_L \cdot g_L \cdot ((f \circ g)^{-1})_L$ can be written as $f_L \cdot (f^{-1})_L \cdot f_R \cdot g_L \cdot ((f \circ g)^{-1})_L$. We must have $f_R \cdot g_L = (f \circ g)_L$ so that

$$f_L \cdot g_L \cdot ((f \circ g)^{-1})_L = f_L \cdot (f^{-1})_L \cdot (f \circ g)_L \cdot ((f \circ g)^{-1})_L.$$

Arguing similarly with right branches gives:

A is generated by words of the form $f_L \cdot (f^{-1})_L$.

We now claim that all words of the form $f_L \cdot (f^{-1})_L$ with $f(\infty) \neq \infty$ are equal in G_-.

Let $g_L \cdot (g^{-1})_L$ be another generator of A. Suppose $(f^{-1})_L(a) = \infty$, $g_R(\infty) = b$, and consider the case where $b \leq a$. Form $f_L \cdot (f^{-1})_L \cdot g_R \cdot (g^{-1})_R$. The product $(f^{-1})_L \cdot g_R$ is equal to $(f^{-1} \circ g)_R$; the product $(f^{-1} \circ g)_R \cdot (g^{-1})_R$ is also defined and equal to $(f^{-1} \circ g \circ g^{-1})_R$ which is $(f^{-1})_R$. Therefore, $f_L \cdot (f^{-1})_L \cdot g_R \cdot (g^{-1})_R$ is the identity as claimed (see Fig. 2).

The argument when $a \leq b$ is similar.

Now, no element of the form $f_L \cdot (f^{-1})_L$ or $f_R \cdot (f^{-1})_R$ with $f(\infty) \neq \infty$ can be the identity since these words are minimal. Furthermore, the n-fold product of $f_L \cdot (f^{-1})_L$ with itself is minimal, as well as the n-fold product of $f_R \cdot (f^{-1})_R$ with itself. Therefore, A is infinite cyclic, and this concludes the proof of the lemma. □

Identifying A with $\mathbb{Z}$ requires a choice. Any $f_L \cdot (f^{-1})_L$ with $f(\infty) \neq \infty$ will represent the negative generator, and any $f_R \cdot (f^{-1})_R$ will represent the positive generator.

We note that the cohomology class of the extension is independent of the choice of the point that plays the role of ∞ since any point can be mapped to ∞. The map induces a conjugation in G and a conjugation always induces the identity in cohomology.

Also note that every element of G_- is a homeomorphism of an open set in $\mathbb{R}$ so has infinite order.

The group G_- seems to be related to the fundamental group Π of the classifying space for codimension 1 real analytic foliations, [1], although we do not have any insight into the meaning of the connection. If G^ω is the group of real analytic orientation preserving homeomorphisms of the circle then G^ω_- is a subgroup of Π.

We refer to the extension as the Discrete Euler Class, although the correspondence with the Classical Euler Class has yet to be made explicit.

Lemma B. *The extension $0 \to \mathbb{Z} \to G_- \to G \to 1$ is non-trivial.*

Proof of Lemma B. We construct a 2-cycle on G which lifts to a 2-chain on G_- whose boundary is -1 in $\mathbb{Z}$.

The cycle will be obtained by writing $x - 6$ in two ways as a commutator in $\mathbb{P}SL(2, (\mathbb{R}) \subset G$.

Define elements of $\mathbb{P}SL(2, (\mathbb{R})$ as follows.

$$f = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \qquad g = \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then

$$fg = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \qquad f^{-1}g^{-1} = \begin{pmatrix} -1 & 5 \\ -1 & 4 \end{pmatrix} \qquad [f, g] = \begin{pmatrix} 1 & -6 \\ 0 & 1 \end{pmatrix}.$$

Now let $F = 7x$, $G = x - 1$, so that, again, $[F, G] = x - 6$.

The product of commutators $[f, g][F, G]^{-1}$ determines a representation of a surface of genus 2 in G, in fact in $PSL(2, \mathbb{R})$.

We lift f to f_R and g to g_L. Since F and G both map ∞ to ∞ their left and right branches are equal. We omit the subscript from the lifts.

The commutator lifts to the word

$$[f_R, g_L] \cdot [F, G]^{-1} = f_R \cdot g_L \cdot (f^{-1})_L \cdot (g^{-1})_R \cdot G \cdot F \cdot G^{-1} \cdot F^{-1}.$$

We have

$$g_L : (-\infty, 0) \to (3, \infty),$$
$$f_R : (2, \infty,) \to (-\infty, -1),$$

with respective inverses

$$(g^{-1})_R : (3, \infty) \to (-\infty, 0)$$
$$(f^{-1})_L : (-\infty, -1) \to (2, \infty).$$

The branches f_R and g_L are composable and their composite is $(fg)_L : (-\infty, 1) \to (-2, \infty)$.

The branches $(f^{-1})_L$ and $(g^{-1})_R$ are composable and their composite is $((gf)^{-1})_L : (-\infty, 4) \to (1, \infty)$.

The word $(fg)_L \cdot ((gf)^{-1})_L$ is minimal.

The word $G \cdot F \cdot G^{-1} \cdot F^{-1}$ is equal to $x + 6$.

Therefore,

$$[f_R, g_L] \cdot [F, G]^{-1} = (fg)_L \cdot ((gf)^{-1})_L \cdot x + 6 = (fg)_L \cdot ((fg)^{-1})_L$$

which represents the negative generator of $\mathbb{Z}$. $\square$

The graphs of the homeomorphisms involved in the construction of the commutators show a remarkable symmetry. See the figures on the following page.

3. The homological algebra approach

This section is concerned with formal simplicial constructions.

To analyze the Euler Class in its various forms we will use an approach that allows us to compare the discrete and continuous groups in homology: group $\to$ group action $\to$ simplicial groupoid $\to$ bisimplicial set $\to$ double abelian group $\to$ spectral sequence $\to$ homology.

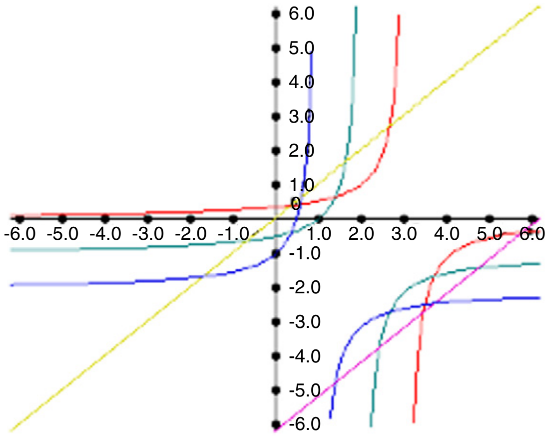
Typically an action will provide, in the context of its associated bisimplicial set, two ways of computing homology depending on the type of simplicial models used. Invariants may appear in the homology of the isotropy groups of the action, or in the homology of the orbits of the action. For G_τ, the topological homeomorphism group, the classical Euler Class appears most naturally in the isotropy of the action. On the other hand, for G_δ, the discrete homeomorphism group, the Discrete Euler Class appears naturally in the orbits of the action. To compare the two in homology we can modify simplices in the continuous case to make the Euler Class appear in the orbits, or modify simplices in the discrete case to make the Euler Class appear in isotropy. Either approach works, and when carried out yields a homology equivalence $BG_\delta \to BG_\tau$.

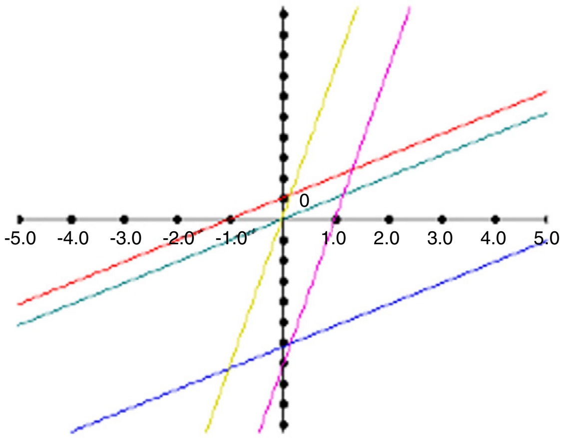
The equivalence is hidden in foliation theory, and is not commonly known. It can be deduced from Thurston's generalization of Mather's Theorem on the loops on the classifying space for Γ_1^0-structures, [10,14]. In our approach we will see explicitly how the Euler Classes, as well as their duals, match up.

Computing the homology of a group

Any group $\mathcal{G}$ acting on a set Θ gives rise to a groupoid $\Gamma\mathcal{G}$ whose objects are Θ, whose morphisms are $\mathcal{G} \times \Theta$, and whose source and target maps are given by $s(g, x) = x$ and $t(g, x) = g(x)$. The composition of morphisms (g, x) and $(f, g(x))$ is (fg, x).

Graphs for Z_1 and Z_2





Z_1: blue is $fg = (2x-1)/(-x+1)$, green is $f = (x-1)/(-x+2)$, red is $g^{-1} = 1/(-x+3)$, yellow is x, pink is $[f,g] = x-6$.

Z_2: blue is $[f,g] = x-6$, $fg = (2x-1)/(-x+1)$, green is x, red is $g^{-1} = x+1$, yellow is $f = 7x$, pink is $fg = 7x-7$.

Consider a topological group $\mathcal{G}$ acting on a space Θ. The group of singular p-simplices $S^p\mathcal{G}$ acts on the set Θ^{p+1}, by $\sigma \cdot (x_0, \ldots, x_p) = (\sigma(0)(x_0), \ldots, \sigma(p)(x_p))$. So $p \to \Gamma S^p\mathcal{G}$ forms a simplicial groupoid. Extending by nerves N in the q-direction produces a bisimplicial set $N\Gamma S\mathcal{G} : (p,q) \to N_q\Gamma S^p\mathcal{G}$, which has $B\mathcal{G}$ as its realization, (up to homotopy), for $p \to \Theta^{p+1}$ has a contractible realization. So realizing the bisimplicial set horizontally and then vertically gives $B\mathcal{G}$.

The simplicial groupoid $p \to \Gamma S^p\mathcal{G}$ will be abbreviated by $\mathcal{G}_*$ and the bisimplicial set $N_q\Gamma S^p\mathcal{G}$ by $\mathcal{G}_{**}$.

The double abelian group $E^0_{p,q}$ of chains on the bisimplicial set gives rise to two spectral sequences converging to $H_{p+q}(B\mathcal{G})$, one obtained by computing homology vertically then horizontally, the other by computing horizontally, then vertically

$$E^2_{p,q} = H^h_p H^v_q(\mathcal{G}_{**}) \Rightarrow H_{p+q}(B\mathcal{G}),$$
$$E^2_{p,q} = H^v_q H^h_p(\mathcal{G}_{**}) \Rightarrow H_{p+q}(B\mathcal{G}).$$

The double abelian group $E^0_{p,q}$ is constructed in two steps. Given $\mathcal{G}_{**}$ let $F^0_{p,q}$ be the free abelian group on the set $\mathcal{G}_{p,q}$, and let $f^h : F_{p,q} \to F_{p-1,q}$ and $f^v : F_{p,q} \to F_{p,q-1}$ be the alternating sum of the face maps. Then, $f^h \circ f^v = f^v \circ f^h$. Now let $E^0_{p,q} = F^0_{p,q}$ but with differentials $d^v = (-1)^p f^v$ and $d^h = f^h$. Then $d^h \circ d^v = -d^v \circ d^h$, and $E^0_{p,q}$ is a double abelian group.

The complex $p \to S^p\mathcal{G}$ may be replaced by a subsimplicial set, or a quotient of a subsimplicial set if the resulting realization is homotopy equivalent to $\mathcal{G}$.

The homotopy groups of $B\mathcal{G}_{**}$ can be computed in the same way if all the simplicial sets in the direction of the first calculation have connected realizations.

Whenever possible, to simplify notation, a single letter, and a double subindex will be used to denote a bisimplicial set. The horizontal complex will be the singular complex, or a variant of it; the vertical complex will be the nerve. Horizontal chain maps will be denoted by d^h and vertical by d^v.

The constructions make sense when $\mathcal{G}$ has the discrete topology. In this case $S^p\mathcal{G} = \mathcal{G}$. The groupoid $p \to \Gamma S^p\mathcal{G} = \mathcal{G}_p$ has simplicial objects $p \to \Theta^{p+1}$, and simplicial morphisms $p \to \mathcal{G} \times \Theta^{p+1}$, and so gives rise to a simplicial groupoid $\mathcal{G}_*$ and bisimplicial set $\mathcal{G}_{**}$ whose realization is $B\mathcal{G}$.

The bisimplicial set $\mathcal{G}_{**}$ can also be obtained as follows. The group $\mathcal{G}$ acts on the infinite simplex Δ^∞ and all its faces, which is a simplicial complex whose p-simplices are $p+1$-element subsets of points of Θ. The action determines, for each p, a discrete groupoid. Extending by nerves in the q direction gives a bicomplex all of whose horizontal complexes are simplicial complexes. Replacing each horizontal simplicial complex by its associated simplicial set gives $\mathcal{G}_{**}$. The realization of a bisimplicial set is a C.W. complex.

We now discuss orbits.

There is a natural map from $B\mathcal{G}$ to the "orbit complex" of the simplicial action.

Consider the bisimplicial set $\mathcal{G}_{**}$, constructed above. Fix p. The vertical complex $\mathcal{G}_{p*}$ is the nerve of a groupoid $\mathcal{G}_p$ with morphisms $\mathcal{G}_{p1}$, and objects $\mathcal{G}_{p0}$.

There is a 1–1 correspondence between the orbits of the action of $\mathcal{G}$ on Θ^{p+1}, and $\pi_0(B\mathcal{G}_p)$, the set of components of $B\mathcal{G}_p$, and the correspondence is simplicial in p. Define $B\mathcal{G}_p \to \pi_0(B\mathcal{G}_p)$ to be the function which maps a given point to the component in which it lies. This produces a map $\pi : B\mathcal{G} \to |\pi_0(B\mathcal{G}_*)|$. Let $orb\mathcal{G}$ denote the space $|\pi_0(B\mathcal{G}_*)|$.

Now we describe the chain complex $C_p(orbG)$ as a quotient of a subcomplex of the chains on the infinite simplex. Let $\sigma = \{\mathbf{0}, \dots, \mathbf{p-1}, \mathbf{p}\}$ denote a set of $p+1$ points of the circle ordered counterclockwise as $\mathbf{0} < \cdots < \mathbf{p-1} < \mathbf{p}$. The non-degenerate p-simplices of the orbit complex are in 1–1 correspondence with the elements of the permutation group $\Pi(p+1) = \Pi\{\mathbf{0}, \dots, \mathbf{p-1}, \mathbf{p}\}$ modulo cyclic permutations. This is the same as $\Pi(p)$, for any set of $p+1$ points can be mapped, as a set, to σ by some element of G, but a $(p+1)$-tuple of elements of σ can be mapped to another $(p+1)$-tuple of elements of σ by an element of G if and only if one of the $(p+1)$-tuples is obtained by cyclically permuting the entries of the other.

So $\Pi(*)$ has the structure of a simplicial set, which we refer to as the *permutation complex*.

4. Comparison of BG_δ and BG_τ

This section is devoted to giving an elementary and constructive proof of the following.

Theorem 2. *The inclusion $G_\delta \to G_\tau$ induces a homology equivalence on classifying spaces. The homology, which is $\mathbb{Z}$ in even dimensions and* 0 *otherwise, is also the homology of the permutation complex $n \to \Pi(n)$. There are isomorphisms*

$$H_p(BG_\tau) \leftarrow H_p(BG_\delta) \to H_p(orbG_\delta) \to H_p(|\Pi(*)|).$$

The positive generator of $H_2(BG_\tau)$ is determined by a counterclockwise loop in S^1 and under the above isomorphisms is identified with the homology class of the 2*-cycle* $[\mathbf{0}, \mathbf{2}, \mathbf{1}]$–$[\mathbf{0}, \mathbf{1}, \mathbf{2}]$ *in the oriented chain complex of $orbG_\delta$.*

The isomorphism can be deduced from the work of Thurston, and Mather on the classifying spaces for foliations. We obtain it directly from bisimplicial constructions and spectral sequences by comparing the groups $G_\delta \to G_\tau \leftarrow S^1_\tau$ by means of their actions on the circle.

In the context of several lemmas the homology of BG_τ and BG_δ are computed individually; then their homology groups are tracked through various correspondences to see explicitly how the discrete and continuous Euler Classes become identified.

Lemma C. *The even dimensional homology groups of BG_τ are all isomorphic to $\mathbb{Z}$, and the odd dimensional are zero. A generator of $H_2(BG_\tau)$ is determined by an orientation of S^1.*

Proof. This is classical. The topological group G_τ is homotopy equivalent to S^1_τ. To compute the homotopy and homology of BS^1_τ most directly consider the bisimplicial set $(G_\tau)_{**}$ associated to G_τ acting on its identity element. Computing homotopy horizontally and then vertically shows that the space BG_τ is a $K(\mathbb{Z}, 2)$, which is complex projective space. The $\mathbb{Z}$ appears as $\pi_1(S^1_\tau)$ in the $E^2_{1,1}$ term. All other terms, except $E^2_{0,0}$, are 0. So the spectral sequence gives a specific isomorphism between $\pi_2(BG)$ and $\pi_1(S^1)$, hence also between $H_2(BG)$ and $H_1(S^1)$.

The universal circle bundle over complex projective space, $0 \to S^1_\tau \to EG_\tau \to BG_\tau \to 0$, gives rise to a Gysin-like sequence and all the differentials are isomorphisms, since EG_τ is contractible.

$$\begin{array}{ccccccc} H_0(BG_\tau) & & H_2(BG_\tau) & & H_4(BG_\tau) & & \\ & \nwarrow & & \nwarrow & & & \cdots \\ H_0(BG_\tau) & & H_2(BG_\tau) & & H_4(BG_\tau). & & \end{array}$$

The leftmost differential is the isomorphism $H_2(BG_\tau) \to H_0(BG_\tau) \otimes H_1(S^1)$, induced on homology by the connecting homomorphism, $\pi_2(BG) \to \pi_1(G)$, of the long exact sequence, in homotopy, of the universal fibration.

The Classical Euler Class $E_\tau : H_2(BG_\tau) \to \mathbb{Z}$ is this differential with a choice of isomorphism $H_1(S^1) \cong \mathbb{Z}$. Classically, for the universal complex line bundle, the isomorphism is determined by choosing the ordered basis, $\{1, i\}$, of the complex plane to correspond to $1 \in \mathbb{Z}$. In the formulation given here, and in all classical formulations, this amounts to choosing the counterclockwise orientation of the circle as corresponding to 1 under the isomorphism $H_1(S^1_\tau) \to \mathbb{Z}$.

The Gysin-like sequence shows that the homology of BG_τ is isomorphic to $\mathbb{Z}$ in each even dimension and 0 otherwise. □

Lemma D. *The even dimensional homology groups of BG_δ are isomorphic to $\mathbb{Z}$, and the odd dimensional are zero. A generator of $H_2(BG_\delta)$ is determined by the 2-cycle $[\mathbf{0}, \mathbf{2}, \mathbf{1}] - [\mathbf{0}, \mathbf{1}, \mathbf{2}]$ in the oriented chain complex of $orbG_\delta$.*

Proof. Consider the double chain complex $E^2_{p,q}$ associated to G_δ acting on Δ^∞. The homology computation that follows will use the fact that the discrete group of orientation preserving homeomorphisms of the reals, $Homeo^+\mathbb{R}$, is acyclic, [9]. Computing homology vertically, then horizontally, gives

$$E^2_{p,q} = \begin{cases} H_p(orbG_\delta) & \text{if } q = 0 \\ 0 & \text{if } q > 0. \end{cases}$$

The isotropy group of each p-simplex is acyclic, for the isotropy group G_b of any point b is isomorphic to the group of orientation preserving homeomorphisms of $S^1 - \{b\}$, which in turn is isomorphic to $Homeo^+\mathbb{R}$. Furthermore, the isotropy group of a $p+1$-element subset of points of S^1 is isomorphic to a $p+1$-fold cartesian product of $Homeo^+\mathbb{R}$, so it too is acyclic. Therefore, the orbit map $BG_\delta \to orbG_\delta$ induces an isomorphism in homology.

For a discrete group action, the bicomplex G_{**} has horizontal simplicial sets which are associated to simplicial complexes. So, in this case, oriented chains can be used. We use oriented chains to compute the homology of the orbit complex. (That this is valid follows from an application of the acyclic model theorem.) All even permutations are identified to a single generator $(\mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots, \mathbf{p})$, and the odd permutations to $-(\mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots, \mathbf{p})$. When p is even cyclic permutations of $p+1$ elements are even, so no further relations are imposed by factoring out by cyclic permutations. Therefore the group of oriented p-chains, when p is even, is isomorphic to $\mathbb{Z}$ and generated by $[\mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots, \mathbf{p}]$. Now when p is odd, $(\mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots, \mathbf{p}) = (\mathbf{1}, \mathbf{2}, \ldots, \mathbf{p}, \mathbf{0})$ since the two $p+1$ tuples are in the same orbit. But $[\mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots, \mathbf{p}] = -[\mathbf{1}, \mathbf{2}, \ldots, \mathbf{p}, \mathbf{0}]$ since cyclic permutations are odd. So $[\mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots, \mathbf{p}] = -[\mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots, \mathbf{p}]$, and the p-th oriented chain group is isomorphic to $\mathbb{Z}_2$. The oriented chain complex of $orbG$ is therefore $0 \leftarrow \mathbb{Z} \leftarrow \mathbb{Z}_2 \leftarrow \mathbb{Z} \leftarrow \mathbb{Z}_2 \leftarrow \cdots$, and the homology is isomorphic to $\mathbb{Z}$ in even dimensions and 0 in odd. We consider $[\mathbf{0}, \mathbf{2}, \mathbf{1}]$–$[\mathbf{0}, \mathbf{1}, \mathbf{2}]$ to be the positive generator of $H_2(BG_\delta.)$

We continue with our analysis of G_τ and G_δ in order to make the homology correspondence explicit. □

The rotation group acting on the circle

Let S^1_τ be identified with the rotation group $U(1)$ acting on the circle. There are two natural ways of extending the actions simplicially, and constructing bicomplexes which realize BG_τ. The first uses simplices which describe multivalued transformations of the circle. The Euler Class appears in the isotropy of the action. The second makes use of the cyclic ordering of points of the circle, and simplices act as single valued transformations of the circle. The Euler Class is in the orbits of this action.

The multivalued action: the isotropy dual Euler Class, $\mathcal{E}_{iso}$

The simplicial model for $U(1)$, $\Delta U(1)$, will be the nerve of the path groupoid. Let $\Delta^1 U(1)$ be the path groupoid on $U(1)$. This is a groupoid in two ways. "Horizontally" it is a groupoid with objects $\Delta^0 U(1)$, the discrete set of rotations. "Vertically" it is a groupoid with objects $S^1_\tau \times S^1_\tau$. Extending by nerves both horizontally and vertically produces the bisimplicial set $U(1)_{**}$ the realization of which is $BU(1)$.

$$\begin{array}{ccc} \Delta^0 U(1) & \overset{\leftarrow}{\leftarrow} & \Delta^1 U(1) \\ \downarrow\downarrow & & \downarrow\downarrow \\ S^1 & \overset{\leftarrow}{\leftarrow} & S^1 \times S^1. \end{array}$$

The path groupoid $\Delta^1 U(1)$ is a quotient groupoid of the full set of singular 1-simplices on $U(1)$. To compose elements vertically choose representatives as singular simplices and compose those point-wise. An element of the path groupoid on $U(1)$ is, in general "multivalued". It is characterized by a beginning rotation in $[0, 2\pi]$, which is its horizontal source, an ending rotation in $[0, 2\pi]$, which is its horizontal target, and a winding number, the number of circuits the path completes as it winds from beginning to end.

An element, σ, of the path groupoid on $U(1)$ is then a family of rotations, continuously parametrized by $[0, 1]$. It acts vertically on ordered pairs of points on the circle, transforming (x, y) which is it vertical source, to $(\sigma(0) \cdot x, \sigma(1) \cdot y)$, which is its vertical target. That is, it rotates x by $\sigma(0)$, and y by $\sigma(1)$.

In general, an element σ of $\Delta^p U(1)$ is formed by concatenating p paths, and so can be viewed as a family of rotations parametrized by $[0, p]$. It acts vertically on $p+1$-tuples of points on the circle, transforming $(x, y, \ldots)$ to $(\sigma(0) \cdot x, \sigma(1) \cdot y, \ldots)$.

Consider $U(1)_{**}$ and the associated double complex of chains E^0. Each vertical complex of $U(1)_{**}$ is connected, for there is a parametrized family of rotations transforming any ordered pair of points on the circle to any other. Furthermore, the isotropy group of a point of S^1 is trivial.

So dual to the Euler Class is a homology class represented by a simplex $\mathcal{E}_{iso} \in E^0_{1,1}$ which considered as a chain, is vertically and horizontally a 1-cycle. Explicitly the isotropy group of a pair (0, 0) consists of paths in $U(1)$ leading from winding by 0 to winding by $2n\pi$, and this group is isomorphic to $\mathbb{Z}$.

The elements of the path groupoid $\Delta^1 U(1)$ can be viewed as multivalued maps. A lift of an element σ of this path groupoid to an element of the path groupoid on $\mathbb{R}$ is a parametrized family of translations, and can be considered, (up to composition on the left and right by an integer translation), as an affine map of the reals. So σ itself is a multivalued "affine homeomorphism of the circle". More generally each morphism of $\Delta^p U(1)$ can be viewed as a multivalued, piecewise linear map of the circle. It acts on $p+1$-tuples of points on the circle, transforming $(x, y, \ldots)$ to $(\sigma(x), \sigma(y), \ldots)$, possibly winding around the circle as it transforms the points. To view the action of σ as that of a multivalued piecewise linear homeomorphism, consider it as mapping the p contiguous multivalued intervals of the circle, $[x, y], [y, z], \ldots$ to the contiguous intervals $[\sigma(x), \sigma(y)], [\sigma(y), \sigma(z)], \ldots$.

The single valued action: the orbit dual Euler class, $\mathcal{E}_{orb}$

Again S^1_τ is identified with the rotation group $U(1)$ acting on the circle, but now the simplicial action is modified in such a way that the Euler Class and its powers appear in the orbits rather than the isotropy. We construct a bicomplex $\tilde{U}(1)_{**}$ which is a sub-bicomplex of $U(1)_{**}$ and homotopy equivalent to it.

Consider again $\Delta^p U(1)$ as a vertical groupoid, and consider an object, $(x_0, x_1, \ldots, x_p)$, in $(S^1)^{p+1}$, with all entries distinct. There is a unique permutation, π of $(1, 2, \ldots, p)$, so that $(x_0, x_{\pi(1)}, \ldots)$ is cyclically ordered in a counterclockwise fashion.

Then $[x_0, x_{\pi(1)}], [x_{\pi(1)}, x_{\pi(2)}], \ldots$ are p contiguous intervals of the circle, the union of which, $[x_0, x_p]$, is single valued. By definition, we let $\sigma \in \Delta^p U(1)$ be a morphism from $(x_0, x_1, \ldots)$ to $(y_0, y_1, \ldots)$ of a new groupoid $\tilde{\Delta}^p U(1)$ if there is a single permutation π which cyclically orders both objects, and if σ then defines a piecewise linear homeomorphism between the contiguous intervals $[x_0, x_{\pi(1)}], [x_{\pi(1)}, x_{\pi(2)}], \ldots$ and $[y_0, y_{\pi(1)}], [y_{\pi(1)}, x_{\pi(2)}], \ldots$. There is now at most one morphism between any two objects.

Adjoining simplicial degeneracies in order to make the horizontal simplicial complexes into simplicial sets, the assignment $p \to \tilde{\Delta}^p U(1)$ defines a simplicial groupoid, and the simplicial objects still realize as a contractible space. Write $\tilde{U}(1)_*$ for the simplicial groupoid and $\tilde{U}(1)_{**}$ for its bisimplicial set, and $\tilde{E}$ for its double chain complex. The horizontal simplicial structure of the simplicial groupoid is no longer than that of the nerve of even a category, since single valued paths cannot, in general, be concatenated and remain single valued. Nevertheless $\tilde{U}(1)_{**}$ is a bicomplex which realizes $BU(1)$, since $p \to \tilde{\Delta}^p U(1)$ still realizes $U(1)$.

Computing homology vertically, then horizontally, all the isotropy groups are trivial, so the homology is in the orbit complex: $BU(1) \to orb\tilde{U}(1)$ is a homotopy equivalence.

We have constructed homotopy equivalences

$$orb\tilde{U}(1) \leftarrow |\tilde{U}(1)_{*,*}| \to |U(1)_{*,*}| \to orbU(1) \leftarrow BU(1).$$

The fundamental difference between the two constructions results from how a $p+1$-tuple $(x_0, x_1, \ldots, x_p)$ of points of the circle is viewed. In the isotropy formulation the Cartesian order of the n-tuple is used, and $[x_0, x_1], \ldots, [x_{p-1}, x_p]$ are considered to be contiguous multivalued intervals. In the orbit formulation the cyclic order of the points is used, determining a single valued interval in the circle, $[x_0, x_{\pi(p)}]$.

Lemma E. *The homology of $BU(1)$ is the homology of the complex $n \to \Pi(n)$, and there is a simplicial isomorphism $orbU(1) \to \Pi(*)$ given by the correspondence:*

$$(x_0, x_1, \ldots, x_n) \to (\pi(\mathbf{0}) = \mathbf{0}, \pi(\mathbf{1}), \ldots, \pi(\mathbf{n})).$$

Proof. As in the discrete case, the non-degenerate n-simplices of the orbit complex are in 1–1 correspondence with the elements of $\Pi(n)$. For there is a morphism joining two $n+1$-tuples $(x_0, \ldots, x_n)$ and $(y_0, \ldots, y_n)$ if and only if there is a homeomorphism mapping x_0 to y_0 preserving the counterclockwise ordering of the remaining points. In particular, there is a morphism from $(x_0, \ldots, x_n)$ to $(\mathbf{0}, \pi(\mathbf{1}), \ldots, \pi(\mathbf{n}))$ for a unique permutation π. $\square$

Consider $\Pi(*)$, and the 2-cycle given by $\mathcal{E}_{orb} = [0, 2, 1)] - [0, 1, 2]$. This cycle is identical to the one constructed in $orbG_\delta$ under the identification with $\Pi(*)$. The last step in the proof of Theorem 1 is the following.

Lemma F. *The cycle $\mathcal{E}_{orb}$ on $orb\tilde{U}(1)$, represents the same homology class as $\mathcal{E}_{iso}$, which is a chain on $U(1)_{**}$, under the homotopy equivalences $orb\tilde{U}(1) \leftarrow \tilde{U}(1)_{**} \to U(1)_{**}$.*

Proof. First identify **0**, **1**, and **2** with three equally spaced points on the circle, say 0, $2\pi/3$ and $4\pi/3$, and lift $\mathcal{E}_{orb}$ to the unoriented chain $(0, 2, 1) - (0, 1, 2) \in \tilde{E}^0_{2,0}$,

The path $\mathcal{E}_{iso}$ can be written as a (horizontal) composite of 3 paths, I, J, and K, where I is a single valued counterclockwise path from 0 to 1, J is a single valued counterclockwise path from 1 to 2, and K is a single valued counterclockwise path from 2 to 0.

The morphism I moves, by a pair of rotations, (0, 1) to (0, 2); the morphism J moves (1, 2) to (2, 1); the morphism K moves (2, 0) to (1, 0).

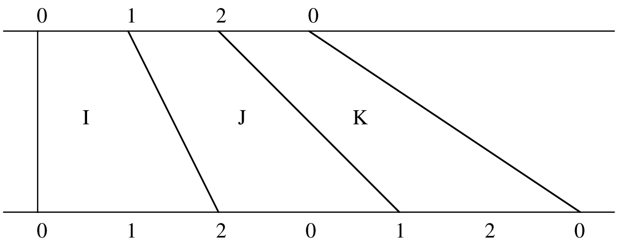


Fig. 3. $\mathcal{E}_{iso}$.

The paths, I, J, and K lie in the subcomplex $|\tilde{U}(1)_{**}| \subset |U(1)_{**}|$. Considered as a chain $I + J + K \in \tilde{E}^0_{1,1}$ has vertical boundary, (recall the vertical differentials on odd indexed columns are negative the alternating sum of the face maps),

$$\begin{aligned}&-[(0,2) + (2,1) + (1,0) - [(0,1) + (1,2) + (2,0)]] =\\&-[(0,2) + (2,1) - (0,1) - [(0,1) + (1,2) - (0,2)]] =\\&-[(2,1) - (01)] + [(0,2) - (1,2)] + [(0,2) - (0,1)],\end{aligned}$$

which is minus the horizontal boundary of $(0, 2, 1) - (0, 1, 2) \in \tilde{E}^0_{2,0}$.

Consider the cycle $I+J+K+[(0,2,1)-(0,1,2)]$ in $\tilde{U}(1)_{**}$. In $U(1)_{**}$, the objects $(0, 2, 1)$ and $(0, 1, 2)$ are in the same groupoid component, so this cycle maps to one that is homologous to $\mathcal{E}_{iso}$. On the other hand, $I+J+K-[(0,2,1)-(0,1,2)]$ maps to $\mathcal{E}_{orb}$ under the quotient map $\tilde{U}(1)_{**} \to orb\tilde{U}(1)$ (see Fig. 3).

This verifies the claim, and establishes a homology isomorphism between BG_τ and BG_δ, which is Theorem 2. □

5. The Discrete Euler Class as a cocycle

Theorem 3. *Let f and g be elements of G, and $0 \in S^1$ a base point. Define*

$$E(f,g) = \begin{cases} 1/2 & \text{if } 0 > f(0) > g(0)\\ -1/2 & \text{if } 0 < f(0) < g(0)\\ 0 & \text{otherwise.}\end{cases}$$

Then E is a 2-cocycle on BG_δ which represents the same cohomology class in $H^2(BG)$ as the one defined by the extension $0 \to \mathbb{Z} \to G_{-} \to G \to 1$.

Proof. Consider $G = G_\delta$, the discrete group of orientation preserving homeomorphisms of the circle, and the orbit complex, $orbG$, obtained from its action on the infinite simplex.

As observed, $BG \to orbG$ is a homology equivalence, and a generator in $H_2(orbG)$ can be represented by the cycle $(\mathbf{0}, \mathbf{2}, \mathbf{1}) - (\mathbf{0}, \mathbf{1}, \mathbf{2})$, where $\mathbf{0} < \mathbf{1} < \mathbf{2}$ are three counterclockwise points of the circle.

To represent the Discrete Euler Class on the orbit complex $orbG$ define a 2-cochain by

$$o(x,y,z) = \begin{cases} 1/2 & \text{if } x > y > z\\ -1/2 & \text{if } x < y < z\\ 0 & \text{otherwise.}\end{cases}$$

To see that o vanishes on boundaries, and is therefore a cocycle, consider

$$\partial(x,y,z,w) = (y,z,w) - (x,z,w) + (x,y,w) - (x,y,z).$$

Suppose $y < z < w$. If x is not on the counterclockwise arc joining y to w then the sign of o evaluated on each of the four simplices forming the boundary is $-$, which means that o on the chain $\partial(x, y, z, w)$ is 0.

If x is strictly between y and z then the sign of o evaluated on each of the last two faces changes to $+$. If x is strictly between z and w, then the sign of o evaluated on the first two faces changes to $+$. If $x = y$ the evaluation on each of the first two faces is $-$ and on the last two is 0. If x is equal to z, the evaluation on the first face is $-$, on the third face is $+$ and on the second and fourth is 0. If x is equal to w, the evaluation on the first and fourth face is $-$ and on each of the middle pair is 0. In any case o on the chain $\partial(x, y, z, w)$ is 0. The exact same argument applies if $y > z > w$, and also if any pair or all three of the points $\{x, y, z\}$ are equal.

To define o in normalized form set x equal to a base point $0 \in S^1$.

$$o(y,z) = \begin{cases} 1/2 & \text{if } 0 > y > z\\ -1/2 & \text{if } 0 < y < z\\ 0 & \text{otherwise.}\end{cases}$$

The class o pulls back to the Discrete Euler Class on BG, as follows.

Let G act on the infinite simplex on the elements of G, Δ_G^∞, as well as on the infinite simplex on the elements of S^1, $\Delta_{S^1}^\infty$.

Let $0 \in S^1$ be a base point. The map $\Delta_G^\infty \to \Delta_{S^1}^\infty$ given $(f, g, \ldots) \to (f(0), g(0), \ldots)$ induces a homotopy equivalence on the orbit complexes of the respective actions. The orbit complex of the action of G on Δ_G^∞ is the nerve of G.

Pulling back o by this map produces the formula in Theorem 3.

We return now to the construction of the Discrete Euler Class as an extension and show that it is the same as the Discrete Euler constructed by the cocycle formula given above.

We write the cycle $[f, g] \cdot [F, G]^{-1}$ as a 2-chain on the nerve of G, and evaluate the Discrete Euler Class e using the cocycle formula.

Consider the 2-chain on the nerve $Z_1 = ([f, g], fg) + (fg, f) - (g^{-1}, f)$. Its boundary is $[f, g]$; Z_1 realizes geometrically as chains on a handle with boundary $[f, g]$. Let $Z_2 = ([F, G], FG) + (FG, F) - (G^{-1}, F)$. Then $Z = Z_1 - Z_2$ is a cycle which corresponds to $[f, g] \cdot [F, G]^{-1}$.

To evaluate E choose a base point; ∞.

E evaluates to 0 on all the simplices of Z_2 and on $([f, g], fg)$ since F, G and $[f, g]$ all fix ∞.

We have $fg(\infty) = -2$ and $f(\infty) = -1$. Since $\infty < -2 < -1$ determines the counterclockwise orientation of the circle $E(fg, f) = -1/2$.

We have $g^{-1}(\infty) = 0$ and $f(\infty) = -1$. Since $\infty > 0 > -1$ determines the clockwise orientation of the circle $E(g^{-1}, f) = +1/2$.

Therefore, $E[Z] = -1$.

Since the evaluations are the same the extension and the cohomology class determined by the cocycle formula both represent the universal Discrete Euler Class. In fact the computation shows that the Discrete Euler Class of $\mathbb{P}SL(2, \mathbb{R})$ is non-trivial.

This completes all aspects of the proof of Theorem 1. □

6. Bounds for the Discrete Euler Class

Consider a circle bundle over a space M with structure group G_τ. The following statements are equivalent, [15]. The structure group can be reduced to a totally disconnected subgroup. The bundle is induced by a homomorphism $\pi_1(M) \to G_\delta$. The bundle is induced by a continuous map $M \to BG_\delta$. There is a foliation of the total space of the bundle with leaves transversal to the fibers.

Since $BG_\delta \to BG_\tau$ is a homology equivalence there is no universal obstruction to reducing a circle bundle with structure group G_τ to a discrete subgroup. Under special circumstances, however, the Discrete Euler Class can only take on certain values.

Theorem (*J. Wood*). *Let M be a closed oriented surface, $[M]$ its fundamental class, and $\chi(M)$ its Euler Characteristic. Let $h : M \to BG_\delta$ be any map. Then $|\langle E, h_*[M]\rangle| \le -\chi(M)$.*

A general version of Wood's Theorem can be deduced from the fact that e can be defined simplicially on the nerve of G_δ, for the magnitude of its value on a 2-simplex is at most $1/2$.

Theorem 4. *Let X be a topological space, and X_* a simplicial set which realizes X. Consider a homology class $[\alpha] \in H_2(X)$ represented by a 2-cycle $\alpha = \Sigma m_i \alpha_i$. Let $\kappa(\alpha) = \Sigma |m_i|$. Let $h : M \to BG_\delta$ be any map. Then $|\langle E, h_*[\alpha]\rangle| \le \kappa(\alpha)/2$.*

This upper bound in this general context is weaker than the one in Wood's theorem. The fundamental class $[M]$ of an oriented surface M of genus $g \ge 0$ can be constructed using $4g - 2$, 2-simplices, which means that Theorem 4 reduces to $| < E, h_*[M] > | \le 2g - 1 = -\chi(M) + 1$ in that case.

The upper bound can be improved for surfaces. We claim that the Discrete Euler cocycle e always vanishes on some pair of 2-simplices. This reduces the number of simplices on which e has value $1/2$ to at most $4g - 4$.

Consider a chain of the form $Z = ([f, g], fg) + (fg, f) - (g^{-1}, f)$ which realizes one of the handles of a closed oriented surface. Suppose f has a fixed point. If that point is chosen as the distinguished point for computing e the last two terms of Z both evaluate to 0. Similarly, if fg has a fixed point then the first two terms are both 0. If $fg(x) = f(x)$ for some x then g has a fixed point at x and again the last two terms are both 0.

So we may assume for all x either $x < f(x) < fg(x)$ or $x < fg(x) < f(x)$ where recall $a < b < c$ means a counterclockwise path starting at a meets b first and then c. Consider $x < f(x) < fg(x)$ and apply f^{-1} to obtain $x < g(x) < f^{-1}(x)$ which is equivalent to $x < f(x) < g^{-1}(x)$. This means that (fg, f) and (g^{-1}, f) both evaluate to $-1/2$ so the last two terms of Z cancel. The same argument applies to the case $x < fg(x) < f(x)$, which proves the claim. So Theorem 4 reduces to Wood's Theorem when X is a closed oriented surface.

7. A brief look at related work, and some comments

Our interest in the Discrete Euler Class stems from its role in the theory of Mapping Class Groups, [12]. As noted in the Introduction, the Based Mapping Class Group of a closed surface of genus g is a subgroup of $G = Homeo^+S^1$ for each $g > 1$. The powers of the Discrete Euler Class exhibit a non-vanishing/vanishing behavior, with torsion appearing at the threshold dimension g. This type of behavior does not occur for the Topological Euler Class, since all powers are already non-zero in the subgroup $U(1)$. In [8], we consider the more general algebraic question of how powers of the Discrete Euler Class behave with respect to a subgroup $H_\delta \subset G_\delta$.

"Boundedness" of the Discrete Euler Class was first observed by J. Milnor in [11] for circle bundles with structure group $SL(2, \mathbb{R})$, and then generalized to circle bundles with structure group $G = Homeo^+S^1$, by Wood, [15]. We note the difference in the $SL(2, \mathbb{R})$, and $Homeo^+S^1$ cases. The bound is weaker in the latter case by a factor of 2.

In [4] Gromov proved that if $\mathcal{G}$ is a real algebraic subgroup of $GL(n, (R))$, then every primary characteristic class of a discrete $\mathcal{G}$-bundle can be represented by a bounded cocycle. Recently Bucher-Karlsson, [1], strengthened Gromov's theorem, by replacing *bounded cocycle* by *cocycle whose set of values on singular simplices is finite*. We note that the "finiteness/boundedness" issues do not arise for discrete $Homeo^+S^1$ bundles because the cocycle that represents the Discrete Euler Class is already finite in the sense that its value on a simplex can be only $+1/2$, 0 or $-1/2$. We refer the reader to [1] for more details about this distinction.

A longstanding question invoked by the work of this paper is what is the relationship between the cohomology of a topological group and the cohomology of the group made discrete. For the group of homeomorphisms of a manifold an answer is provided by the Thurston–Mather theorem, reproved here in the case of a circle. For a Lie Group, or in its original form, a complex reductive algebraic group, there is the Friedlander–Milnor Conjecture, as yet unproved. See [3], for the current status of the problem.

The indirect nature of the proof of Theorem 2 suggests that it is difficult to relate the Discrete Euler Class, as constructed by the extension of Theorem 1, to the Topological Euler Class. Is there perhaps a way to compare the extension $0 \to \mathbb{Z} \to G_- \to G_\delta \to 1$ directly to $0 \to \mathbb{Z} \to Homeo^+\mathbb{R}_\tau \to G_\tau \to 1$? If so, maybe the powers of the extensions can be compared as well giving a more intrinsic homological algebra proof of Theorem 2.

The non-vanishing $\to$ torsion threshold $\to$ vanishing behavior of the powers of the Euler Class of the Based Mapping Class Groups suggests that powers of cohomology classes pulled back to discrete subgroups should in some way be accounted for in a theory relating continuous and discrete groups. Perhaps some insight would be gained by constructing powers of the extension, $0 \to \mathbb{Z} \to H_- \to H \to 0$, for $H \subset G$, and studying its properties.

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